ON THE REDUCTION OF CERTAIN DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

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It is proposed to investigate in this paper the reduction of the differential equations of the second order

$$(1) \qquad \begin{aligned} x_1' &= +p_1 x_1 + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij} x_1^i x_2^j = +p_1 x_1 + X_1, \\ x_2' &= -p_2 x_2 + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} b_{ij} x_1^i x_2^j = -p_2 x_2 + X_2 \qquad (i+j \geqq 2), \end{aligned}$$

where p_1 and p_2 are two positive numbers, and the coefficients a_{ij}, b_{ij} are constants, to similar differential equations in which the right members are polynomials, by means of the linear-transcendental substitution

$$(2) \qquad \begin{aligned} x_1 &= y_1 + \sum\sum c_{ij} y_1^i y_2^j, \\ x_2 &= y_2 + \sum\sum d_{ij} y_1^i y_2^j \qquad (i+j \geqq 2). \end{aligned}$$

Two essentially distinct cases present themselves: first, the ratio p_1/p_2 is irrational; second, the ratio p_1/p_2 is rational. It will be shown in the case $p_1/p_2 = \beta$ is irrational and the number β satisfies a rather mild condition that there exists a convergent transformation which reduces the differential equations to their linear terms. If the ratio is rational and the equations are canonical a convergent transformation exists for which the reduced differential equations have the form

$$\begin{aligned} y_1' &= +y_1(1 + Q\, y_1 y_2), \\ y_2' &= -y_2(1 + Q\, y_1 y_2), \end{aligned}$$

which are easily integrated. If the ratio is rational and the equations are not canonical then it is always possible to reduce the differential equations to algebraic forms but nothing can be said as to the convergence of the transformations without further knowledge of the coefficients of equations (1),

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except that divergent transformations are quite possible. A better statement, perhaps, would be that further knowledge of the coefficients is required before suitable algebraic forms can be determined. It seems certain that a knowledge of a few coefficients of terms of low degree is not sufficient.

Before entering upon the general discussion we shall show that it is no essential restriction upon the differential equations to suppose that x_1 is a factor of the right member of the first equation of (1) and that x_2 is a factor of the second equation; for if it were not so, and we take

$$(3)\qquad \begin{aligned} x_1 &= y_1 + f_2(y_2),\\ x_2 &= y_2 + f_1(y_1) \end{aligned}$$

then $f_1(y_1)$ and $f_2(y_2)$ can be so chosen that the resulting equations in y_1 and y_2 will have this property. Making the substitution (3) in (1) and solving the resulting equations there is obtained

$$(4)\qquad \begin{aligned} \left(1 - \frac{df_1}{dy_1}\frac{df_2}{dy_2}\right) y_1' &= + [p_1(y_1 + f_2) + X_1] + [p_2(y_2 + f_1) - X_2]\frac{df_2}{dy_2},\\ \left(1 - \frac{df_1}{dy_1}\frac{df_2}{dy_2}\right) y_2' &= - [p_2(y_2 + f_1) - X_2] - [p_1(y_1 + f_2) + X_1]\frac{df_1}{dy_1}. \end{aligned}$$

The right members of these equations will carry y_1 and y_2 respectively as factors if the conditions

$$(5)\qquad \begin{aligned} &[p_1 y_1 + X_1(y_1, f_1)]\frac{df_1}{dy_1} + [p_2 f_1 - X_2(y_1, f_1)] = 0,\\ &[p_2 y_2 - X_2(f_2, y_2)]\frac{df_2}{dy_2} + [p_1 f_2 + X_1(f_2, y_2)] = 0 \end{aligned}$$

are satisfied. These conditions are obtained by putting $y_1 = 0$ in the right member of the first equation and $y_2 = 0$ in the right member of the second equation. The two equations of (5) are of the same type, the first depending only on the variable y_1 and the second only on the variable y_2. The solution of either is dominated by the solution of

$$(5')\qquad \left[y - \frac{M(y+f)^2}{1 - \alpha(y+f)}\right]\frac{df}{dy} + \beta f = \frac{M(y+f)^2}{1 - \alpha(y+f)}$$

where M and α are positive constants suitably chosen, and $\beta = p_1/p_2$ or its reciprocal. If we take $f = y\phi$ it is found that the solution of (5′) is dominated by the solution of

$$(5'')\qquad y\frac{\partial\phi}{\partial y} + (1 + \beta)\phi = \frac{My(1 + \beta\phi)(1 + \phi)^2}{1 - \alpha y(1 + \phi) - My(1 + \phi)^2},$$

and the solution of (5″) is dominated by the solution of

$$(1+\beta)\phi = \frac{My(1+\beta\phi)(1+\phi)^2}{1-\alpha y(1+\phi) - My(1+\phi)^2}$$

which is convergent, whatever may be the value of β, since it is always positive.

If therefore f_1 and f_2 satisfy conditions (5) equations (4) are of the same form as (1) except that y_1 is a factor of the right member of the first equation and y_2 is a factor of the right member of the second. It is therefore no essential restriction to assume that the differential equations are of this form.

I. The ratio p_1/p_2 is irrational

Let us suppose that p_1/p_2 is irrational and equal to γ. Then the equations can be written

$$\text{(6)} \qquad \begin{aligned} x_1' &= x_1(+1+X_1), \\ x_2' &= x_2(-\gamma+X_2), \end{aligned}$$

where X_1 and X_2 are power series in x_1 and x_2 of order one. We make now the substitution

$$\text{(7)} \qquad x_1 = y_1(1+f_1) = y_1 e^{g_1}, \qquad x_2 = y_2(1+f_2) = y_2 e^{g_2},$$

where f_1, f_2, g_1, g_2 are power series in y_1 and y_2 vanishing for $y_1 = y_2 = 0$. If we assume that

$$\text{(8)} \qquad y_1' = y_1, \qquad y_2' = -\gamma y_2,$$

we obtain

$$\text{(9)} \qquad \begin{aligned} y_1\frac{\partial g_1}{\partial y_1} - \gamma y_2\frac{\partial g_1}{\partial y_2} &= X_1(y_1 e^{g_1}, y_2 e^{g_2}), \\ y_1\frac{\partial g_2}{\partial y_2} - \gamma y_2\frac{\partial g_2}{\partial y_2} &= X_2(y_1 e^{g_1}, y_2 e^{g_2}). \end{aligned}$$

If we set

$$\text{(10)} \qquad \begin{aligned} g_1 &= \sum\sum g_{ij}^{(1)} y_1^i y_2^j, \\ g_2 &= \sum\sum g_{ij}^{(2)} y_1^i y_2^j, \end{aligned}$$

we find for the determination of the coefficients $g_{ij}^{(1)}$ and $g_{ij}^{(2)}$ the relations

$$\begin{aligned} (i-j\gamma)g_{ij}^{(1)} &= G_{ij}^{(1)}, \\ (i-j\gamma)g_{ij}^{(2)} &= G_{ij}^{(2)}, \end{aligned}$$

where at each step $G_{ij}^{(1)}$ and $G_{ij}^{(2)}$ are known quantities. Since γ is irrational none of the coefficients $(i-j\gamma)$ vanishes and there is no difficulty in determining the coefficients $g_{ij}^{(1)}$ and $g_{ij}^{(2)}$ which define the transformation.

It follows from (9) and the fact that g_1 and g_2 are power series in y_1 and y_2

of order one that $g_1 = g_2$ if $X_1 = X_2$. For the question of convergence therefore it will be sufficient to consider the single equation

$$y_1 \frac{\partial g}{\partial y_1} - \gamma y_2 \frac{\partial g}{\partial y_2} = \left[\frac{1}{1 - y_1} \cdot \frac{1}{1 - y_2} - 1 \right] \frac{M}{1 - g}, \tag{11}$$

which dominates (9). Since γ occurs with the negative sign in the left member of this equation the solution does not have all of its signs positive and one could not be sure that the solution of (11) dominates the solution of (9). Let us consider first the solution of

$$y_1 \frac{\partial g}{\partial y_1} + \gamma y_2 \frac{\partial g}{\partial y_2} = \left[\frac{1}{1 - y_1} \cdot \frac{1}{1 - y_2} - 1 \right] \frac{M}{1 - g}. \tag{12}$$

The solution of this equation has all of its terms positive and no cancellation occurs, and the coefficients so far as they depend upon the right member of (12) have maximum values. The solution of (12) is easily obtained, for if we take

$$\omega = g - \tfrac{1}{2} g^2$$

we get

$$y_1 \frac{\partial \omega}{\partial y_1} + \gamma y_2 \frac{\partial \omega}{\partial y_2} = M \sum \sum y_1^i y_2^j$$

and consequently

$$\omega = M \sum \sum \frac{y_1^i y_2^j}{i + j\gamma}.$$

Since $g = 1 - \sqrt{1 - 2\omega}$ is expansible in powers of ω with positive coefficients it is easy to see how the function g is built up and all of the coefficients so far as they depend upon the right member of (12) have maximum values.

If now we change γ into $-\gamma$ the solution of (12) becomes the solution of (11). We have

$$g = 1 - \sqrt{1 - 2\omega},$$

$$\omega = M \sum \sum \frac{y_1^i y_2^j}{i - j\gamma}.$$

Since the series

$$\sum \sum \frac{y_1^i y_2^j}{|i - j\gamma|}$$

is convergent for a certain class of irrational numbers*, γ, it follows that if we take

$$\omega^* = M \sum \sum \frac{y_1^i y_2^j}{|i - j\gamma|}, \qquad g^* = 1 - \sqrt{1 - 2\omega^*}$$

*See Bulletin of the American Mathematical Society, vol. 22 (1915), p. 26–32.

then g^* will be a convergent series which will dominate the solutions (9) for the coefficients of g^* will be a maximum in so far as they depend upon the right members through M and a maximum in so far as they depend upon γ being negative.

It follows therefore, since $y_1 = \mu_1 e^t$, $y_2 = \mu_2 e^{-\gamma t}$, that x_1 and x_2 are expansible in powers of $\mu_1 e^t$ and $\mu_2 e^{-\gamma t}$, and that these expansions are convergent for any preassigned value of t provided μ_1 and μ_2 are sufficiently small.

II. The ratio p_1/p_2 is rational

If the ratio p_1/p_2 is rational it is always possible by a linear change of the independent variable, t, to make p_1 and p_2 integers, and we will suppose this to have been done. Then by the substitution

$$x_1 = \xi_1^{p_1}, \qquad x_2 = \xi_2^{p_2},$$

we will have, since

$$\frac{1}{p_1}\frac{x_1'}{x_1} = \frac{\xi_1'}{\xi_1}, \qquad \frac{1}{p_2}\frac{x_2'}{x_2} = \frac{\xi_2'}{\xi_2},$$

two equations in ξ_1 and ξ_2 of the same type as (1), in which p_1 and p_2 are each unity. We may therefore suppose $p_1 = p_2 = 1$, and the equations are

$$\begin{aligned} x_1' &= x_1(+1 + X_1^{(1)}), \\ x_2' &= x_2(-1 + X_2^{(2)}). \end{aligned} \tag{13}$$

For our present purposes, however, it will be preferable to use the less specialized form

$$\begin{aligned} x_1' &= +x_1 + X_1, \\ x_2' &= -x_2 + X_2, \end{aligned} \tag{13}$$

where

$$X_1 = \sum\sum a_{ij} x_1^i x_2^j, \qquad X_2 = \sum\sum b_{ij} x_1^i x_2^j \qquad (i + j \geqq 2),$$

in which it is not supposed that x_1 is a factor of X_1 nor x_2 a factor of X_2.

If, in (13), we make the linear-transcendental substitution

$$\begin{aligned} x_1 &= y_1 + \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} c_{ij} y_1^i y_2^j = y_1 + f_1(y_1, y_2), \\ x_2 &= y_2 + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_{ij} y_1^i y_2^j = y_2 + f_2(y_1, y_2) \end{aligned} \qquad (i + j \geqq 2), \tag{14}$$

it is found at once that the differential equations cannot be reduced to their linear terms, for it is not possible, in general, to determine the constants c_{ij} and d_{ij} so as to do this. It is, however, always possible to reduce them to the form

$$
(15) \qquad \begin{aligned} y_1' &= + y_1 + Q_1^{(1)} y_1^2 y_2 + Q_1^{(2)} y_1^3 y_2^2, \\ y_2' &= - y_2 + Q_2^{(1)} y_1 y_2^2 + Q_2^{(2)} y_1^2 y_2^3, \end{aligned}
$$

in which $Q_i^{(j)}$ are constants provided $Q_1^{(1)} + Q_2^{(1)} \neq 0$. On substituting (14) and (15) in (13) there results

$$
(16) \qquad \begin{aligned} \sum_{i,j} (i - j - 1) c_{ij} y_1^i y_2^j &= \sum a_{ij} (y_1 + f_1)^i (y_2 + f_2)^j - Q_1^{(1)} y_1^2 y_2 \\ &\quad - Q_1^{(2)} y_1^3 y_2^2 - [Q_1^{(1)} y_1^2 y_2 + Q_1^{(2)} y_1^3 y_2^2] \frac{\partial f_1}{\partial y_1} \\ &\quad - [Q_2^{(1)} y_1 y_2^2 + Q_2^{(2)} y_1^2 y_2^3] \frac{\partial f_1}{\partial y_2}, \\ \sum (i - j + 1) d_{ij} y_1^i y_2^j &= \sum b_{ij} (y_1 + f_1)^i (y_2 + f_2)^j - Q_2^{(1)} y_1 y_2^2 \\ &\quad - Q_2^{(2)} y_1^2 y_2^3 - [Q_1^{(1)} y_1^2 y_2 + Q_1^{(2)} y_1^3 y_2^2] \frac{\partial f_2}{\partial y_1} \\ &\quad - [Q_2^{(1)} y_1 y_2^2 + Q_2^{(2)} y_1^2 y_2^3] \frac{\partial f_2}{\partial y_2}. \end{aligned}
$$

From these equations it is seen that the c_{ij} are readily determined except for such values of i and j that $i - j = +1$ and that the d_{ij} are readily determined except when $i - j = -1$. Thus, up to and including terms of the second degree,

$$
\begin{aligned} x_1 &= y_1 + [a_{20} y_1^2 - a_{11} y_1 y_2 - \tfrac{1}{3} a_{02} y_2^2], \\ x_2 &= y_2 + [\tfrac{1}{3} b_{20} y_1^2 + b_{11} y_1 y_2 - b_{02} y_2^2]. \end{aligned}
$$

Using these values it is found that the terms of third degree in the right members of (16) become

$$
\begin{aligned} R_1^{(3)} = {} & (2a_{20}^2 + \tfrac{1}{3} a_{11} b_{20} + a_{30}) y_1^3 \\ & + (- a_{20} a_{11} + a_{11} b_{11} + \tfrac{2}{3} a_{02} b_{20} + a_{21} - Q_1^{(1)}) y_1^2 y_2 \\ & + (- \tfrac{2}{3} a_{20} a_{02} - a_{11}^2 - a_{11} b_{02} + 2 a_{02} b_{11} + a_{12}) y_1 y_2^2 \\ & + (- \tfrac{1}{3} a_{11} a_{02} - 2 a_{02} b_{02} + a_{03}) y_1^3, \\ R_2^{(3)} = {} & (2 a_{20} b_{20} + \tfrac{1}{3} b_{20} b_{11} + b_{30}) y_1^3 \\ & + (- 2 b_{20} a_{11} + b_{11} a_{20} + \tfrac{2}{3} b_{20} b_{02} + b_{21}) y_1^2 y_2 \\ & + (- \tfrac{2}{3} b_{20} a_{02} - a_{11} b_{11} + b_{11} b_{02} + b_{12} - Q_2^{(1)}) y_1 y_2^2 \\ & + (- \tfrac{1}{3} b_{11} a_{02} - 2 b_{02}^2 + b_{03}) y_2^3. \end{aligned}
$$

Since in the left members of (16) the coefficients of c_{21} and d_{12} are zero these quantities cannot be determined and they therefore remain arbitrary. From the right members therefore we must have

$$
\begin{aligned}
Q_1^{(1)} &= - a_{20} a_{11} + a_{11} b_{11} + \tfrac{2}{3} a_{02} b_{20} + a_{21}, \\
Q_2^{(1)} &= - \tfrac{2}{3} b_{20} a_{02} - a_{11} b_{11} + b_{11} b_{02} + b_{12}, \\
Q_1^{(1)} + Q_2^{(1)} &= - a_{20} a_{11} + b_{11} b_{02} + a_{21} + b_{12},
\end{aligned} \tag{17}
$$

and then we obtain

$$
\begin{aligned}
c_{30} &= a_{20}^2 + \tfrac{1}{6} a_{11} b_{20} + \tfrac{1}{2} a_{30}, \\
c_{21} &= \alpha_1 = \text{arbitrary}, \\
c_{12} &= \tfrac{1}{3} a_{20} a_{02} + \tfrac{1}{2} a_{11}^2 + \tfrac{1}{2} a_{11} b_{02} - a_{02} b_{11} - \tfrac{1}{2} a_{12}, \\
c_{03} &= \tfrac{1}{12} a_{11} a_{02} + \tfrac{1}{2} a_{02} b_{02} - \tfrac{1}{4} a_{03}, \\
d_{30} &= \tfrac{1}{2} a_{20} b_{20} + \tfrac{1}{12} b_{20} b_{11} + \tfrac{1}{4} b_{30}, \\
d_{21} &= - b_{20} a_{11} + \tfrac{1}{2} b_{11} a_{20} + \tfrac{1}{2} b_{11}^2 + \tfrac{1}{3} b_{20} b_{02} + \tfrac{1}{2} b_{21}, \\
d_{12} &= \beta_1 = \text{arbitrary}, \\
d_{03} &= \tfrac{1}{6} b_{11} a_{02} + b_{02}^2 - \tfrac{1}{2} b_{03}.
\end{aligned}
$$

One can continue the computation as far as is desired except for the terms $y_1^{k+1} y_2^k$ in the first equation and $y_1^k y_2^{k+1}$ in the second. The coefficients $c_{k+1,\,k}$ $d_{k,\,k+1}$ (which for brevity we shall denote by α_k and β_k) cannot be determined as are the remaining coefficients. These coefficients which are left arbitrary in the terms of degree $2k + 1$ enter also in terms of higher degree and are determined by the condition that in the right members of (16) the coefficients of the terms $y_1^{k+2} y_2^{k+1}$ in the first equation and $y_1^{k+1} y_2^{k+2}$ in the second equation must vanish. Thus it is found from the coefficients of $y_1^3 y_2^2$ and $y_1^2 y_2^3$ that

$$
\begin{aligned}
Q_1^{(2)} &= - Q_2^{(1)} \alpha_1 + Q_1^{(1)} \beta_1 + \text{known terms}, \\
Q_2^{(2)} &= + Q_2^{(1)} \alpha_1 - Q_1^{(1)} \beta_1 + \text{known terms}.
\end{aligned} \tag{18}
$$

Since the determinant of the coefficients of α_1 and β_1 in these two expressions is zero, α_1 and β_1 cannot in general be chosen so as to make $Q_1^{(2)} = 0$, $Q_2^{(2)} = 0$. It is necessary therefore to retain these constants in the differential equations. But $k = 1$ happens to be exceptional. For a general value of k these two conditions are

$$
\begin{aligned}
- [(k-1) Q_1^{(1)} + k Q_2^{(1)}] \alpha_k + \qquad\qquad Q_1^{(1)} \beta_k &= \text{known terms}, \\
Q_2^{(1)} \alpha_k - [k Q_1^{(1)} + (k-1) Q_2^{(1)}] \beta_k &= \text{known terms}.
\end{aligned} \tag{18$'$}
$$

The value of the determinant $D^{(k)}$ of the coefficients of α_k and β_k is

$$
D^{(k)} = k(k-1)(Q_1^{(1)} + Q_2^{(1)})^2.
$$

If therefore $Q_1^{(1)} + Q_2^{(1)} \neq 0$ the determinant vanishes only if $k = 0$ or $k = 1$, and there is no further difficulty in the determination of the coefficients. The determination is unique aside from the fact that α_1 and β_1 remain arbitrary. Since α_1 and β_1 enter $Q_1^{(2)}$ and $Q_2^{(2)}$ linearly, as is seen from (18), it will be convenient to impose the relations

$$Q_1^{(2)} = mQ_1^{(1)}, \qquad Q_2^{(2)} = mQ_2^{(1)},$$

which is always possible if $Q_1^{(1)} + Q_2^{(1)} \neq 0$, for on adding the two equations of (18) it is found that m equals the sum of the known terms divided by $Q_1^{(1)} + Q_2^{(1)}$. The quantity m is independent of α_1 and β_1 and depends only upon the coefficients of the original differential equations.

With these determinations therefore the differential equations take the form

$$\text{(19)} \qquad \begin{aligned} y_1' &= y_1[+1 + Q_1^{(1)}(1 + my_1 y_2) y_1 y_2], \\ y_2' &= y_2[-1 + Q_2^{(1)}(1 + my_1 y_2) y_1 y_2], \end{aligned}$$

which can be integrated. On taking

$$y_1 = z_1 e^t, \qquad y_2 = z_2 e^{-t},$$

the differential equations become

$$\text{(20)} \qquad \begin{aligned} z_1' &= Q_1^{(1)} z_1 [z_1 z_2 + m z_1^2 z_2^2], \\ z_2' &= Q_2^{(1)} z_2 [z_1 z_2 + m z_1^2 z_2^2]. \end{aligned}$$

From these equations we get

$$\text{(21)} \qquad (z_1 z_2)' = (Q_1^{(1)} + Q_2^{(1)}) z_1 z_2 [z_1 z_2 + m z_1^2 z_2^2],$$

whence

$$\frac{1}{Q_1^{(1)}} \frac{z_1'}{z_1} = \frac{1}{Q_2^{(1)}} \frac{z_2'}{z_2} = \frac{1}{(Q_1^{(1)} + Q_2^{(1)})} \frac{(z_1 z_2)'}{(z_1 z_2)},$$

so that

$$z_1 = c\,(z_1 z_2)^{Q_1^{(1)}/(Q_1^{(1)} + Q_2^{(1)})}, \qquad z_2 = \frac{1}{c}\,(z_1 z_2)^{Q_2^{(1)}/(Q_1^{(1)} + Q_2^{(1)})},$$

$$z_1^{Q_2^{(1)}} = c^2 z_2^{Q_1^{(1)}}.$$

Using the function $L'(\tau)$ which was introduced in the discussion of the differential equation of the first order* it is seen from (21) that, if $m \neq 0$,

$$z_1 z_2 = \frac{L'(\tau) - 1}{m},$$

where

$$\tau = 1 - \frac{Q_1^{(1)} + Q_2^{(1)}}{m}(t - t_0).$$

* See Annals of Mathematics, vol. 19 (1917), p. 21–29.

But if $m = 0$ we have

$$z_1 z_2 = \frac{1}{(Q_1^{(1)} + Q_2^{(1)})(t_0 - t)}.$$

Consequently, if $m \neq 0$, the expressions for y_1 and y_2 are

$$(22)\qquad \begin{aligned} y_1 &= ce^t\left(\frac{L'(\tau) - 1}{m}\right)^{Q_1^{(1)}/(Q_1^{(1)}+Q_2^{(1)})}, \\ y_2 &= \frac{1}{c}e^{-t}\left(\frac{L'(\tau) - 1}{m}\right)^{Q_2^{(1)}/(Q_1^{(1)}+Q_2^{(1)})}. \end{aligned}$$

But if $m = 0$ then

$$(23)\qquad \begin{aligned} y_1 &= \frac{ce^t}{[(Q_1^{(1)} + Q_2^{(1)})(t_0 - t)]^{Q_1^{(1)}/(Q_1^{(1)}+Q_2^{(1)})}}, \\ y_2 &= \frac{\frac{1}{c}e^{-t}}{[(Q_1^{(1)} + Q_2^{(1)})(t_0 - t)]^{Q_2^{(1)}/(Q_1^{(1)}+Q_2^{(1)})}}. \end{aligned}$$

It is to be observed also that equations (19) admit the integral

$$(24)\qquad H = y_1^{-(Q_2^{(1)}+m)}\, y_2^{(Q_1^{(1)}-m)}\,(1 + my_1 y_2)^m\, e^{-1/y_1y_2},$$

or, in logarithmic form,

$$(25)\qquad \begin{aligned} H_1 = (m + Q_2^{(1)}) \log y_1 + (m - Q_1^{(1)}) \log y_2 \\ - m \log (1 + my_1 y_2) + \frac{1}{y_1 y_2}. \end{aligned}$$

The following example is illustrative of this transformation

$$(26)\qquad x_1' = x_1 + x_1^2 + x_1 x_2, \qquad x_2' = - x_2.$$

The transformation defined by the relations

$$(27)\qquad x_1 = \frac{y_1 e^{-y_2}}{1 + y_1 y_2 \int \frac{e^{-y_2} + y_2}{y_2^2}\, dy_2}, \qquad x_2 = y_2,$$

transforms the differential equations into

$$y_1' = y_1(1 - y_1 y_2), \qquad y_2' = - y_2,$$

which have the form of (19) in which $Q_1^{(1)} = -1$, $m = Q_2^{(1)} = 0$, and it will be observed that the transformation considered as a power series in y_1 and y_2 is convergent.

It is not necessary however that the reduced equations shall have the form of (19). Indeed, it is evident from the nature of the process that any desired terms, of degree not less than two, could be added to the right members of (19) and the process of determining the transformation would not fail. Thus it is possible to have an infinite variety of transformations and reduced differential equations, which shows clearly that all of these transformations cannot be convergent. Consider for example, the equations

$$\begin{aligned} x_1' &= x_1[+1 + (1 + \alpha i_1) x_1 x_2 + \mu x_1^{i_2+1} x^{i_1+1}], \\ x_2' &= x_2[-1 - (1 + \alpha i_2) x_1 x_2 - \mu x_1^{i_2+1} x^{i_1+1}], \end{aligned} \tag{28}$$

in which α and μ are any constants and i_1, i_2 are positive integers or zero, but not equal. Here $Q_1^{(1)} + Q_2^{(1)} = \alpha(i_1 - i_2)$ which is not zero if α is not zero. Consequently one can determine a transformation which will reduce (28) to (19). Equations (28) however admit the integral

$$I = x_1^{-i_2} x_2^{-i_1} s e^{s^\alpha/\alpha} + \mu \int e^{s^\alpha/\alpha} ds, \tag{29}$$

where $s^a = 1/x_1 x_2$. If now $\mu \neq 0$ the singularities of I in the neighborhood of the origin are not the same as the singularities of (24). Since $I(y_1 y_2)$ is not a function of $H(y_1 y_2)$, their jacobian being distinct from zero, it is clear that I cannot be transformed into H by a convergent linear-transcendental substitution, which cannot alter the character of the singularities at the origin.* We conclude therefore that the transformation in this case is divergent if μ is distinct from zero. Furthermore, since i_1 and i_2 may be as large as we please we must conclude that the singularities of the integral in the neighborhood of the origin do not depend merely upon the coefficients of the terms of low degree in the differential equations. It would seem to be a reasonable conjecture that, if the differential equations admit an integral which is known, then there exists a convergent linear-transcendental transformation which reduces the differential equations to an algebraic type such that the reduced equations admit an integral having the same singularities at the origin as the original integral, but this is not always true, as is shown by the following example. The equations

$$\begin{aligned} y_1' &= \frac{y_1}{1 + y_1}(+1 + y_1 - y_1 y_2), \\ y_2' &= \frac{y_2}{1 + y_1}(-1 - y_1 - y_1 y_2) \end{aligned} \tag{A}$$

admit the integral

$$H_1 = y_2 s e^{s^2/2} + \int e^{s^2/2} ds, \qquad \text{where} \qquad s^2 = \frac{1}{y_1 y_2}.$$

* Dulac has considered an example in which he has shown from the law of the coefficients that the transformation in that case is divergent. See Bulletin des sciences mathématiques, vol. 37 (1913).

Likewise the equations

$$(B)\qquad \begin{aligned} x_1' &= x_1(+1 + x_1 - x_1 x_2), \\ x_2' &= x_2(-1 - x_1 - x_1 x_2) \end{aligned}$$

admit the integral

$$H_2 = x_2 \sigma e^{\sigma^2/2} + \int e^{\sigma^2/2} d\sigma, \qquad \text{where} \qquad \sigma^2 = \frac{1}{x_1 x_2}.$$

But (A) cannot be transformed into (B) by a linear-transcendental substitution, for we have

$$\frac{y_1'}{y_1} - \frac{y_2'}{y_2} = 2, \qquad \frac{(x_1 x_2)'}{(x_1 x_2)^2} = -2 \qquad \text{so that} \qquad \frac{y_1'}{y_1} - \frac{y_2'}{y_2} = -\frac{(x_1 x_2)'}{(x_1 x_2)^2}.$$

That is $y_1/y_2 = Ce^{1/x_1 x_2}$. Obviously y_1/y_2 cannot be transformed into $Ce^{1/x_1 x_2}$ by a linear transcendental substitution.

Existence of an integral of the form $H = \dfrac{P_1(x_1, x_2)}{P_2(x_1, x_2)}$

It will be shown in the present section, that if there exists an integral of the form

$$H = \frac{P_1(x_1, x_2)}{P_2(x_1, x_2)},$$

where P_1 and P_2 are ordinary power series (and this hypothesis includes the case where the differential equations are canonical) then the transformation is convergent and the reduced equations are

$$(30)\qquad \begin{aligned} y_1' &= y_1(+1 + Qy_1 y_2), \\ y_2' &= y_2(-1 - Qy_1 y_2). \end{aligned}$$

We have already observed that it is always possible to reduce the differential equations to the form

$$(31)\qquad \begin{aligned} y_1' &= y_1(+1 + Q_1^{(1)} y_1 y_2 + Q_1^{(1)} y_1^2 y_2^2 + Q_1^{(3)} y_1^3 y_2^3 + \cdots), \\ y_2' &= y_2(-1 + Q_2^{(1)} y_1 y_2 + Q_2^{(2)} y_1^2 y_2^2 + Q_2^{(3)} y_1^3 y_2^3 + \cdots), \end{aligned}$$

whether $Q_1^{(1)} + Q_2^{(1)} = 0$ or not zero, the $Q_i^{(j)}$ being suitably chosen constants.

On differentiating $H = P_1/P_2$ we have

$$(32)\qquad P_2\left[\frac{\partial P_1}{\partial x_1} x_1' + \frac{\partial P_1}{\partial x_2} x_2'\right] - P_1\left[\frac{\partial P_2}{\partial x_1} x_1' + \frac{\partial P_2}{\partial x_2} x_2'\right] \equiv 0.$$

Let us suppose now that $P_1 = p_1^{(n)} + p_1^{(n+1)} + \cdots$, $P_2 = p_2^{(m)} + p_2^{(m+1)} + \cdots$, where $p_j^{(k)}$ is a polynomial homogeneous in x_1 and x_2 of degree k. Consider the terms of lowest degree, $m + n$, in (32). We have

$$(33) \qquad p_2^{(m)}\left(x_1\frac{\partial p_1^{(n)}}{\partial x_1} - x_2\frac{\partial p_1^{(n)}}{\partial x_2}\right) - p_1^{(n)}\left(x_1\frac{\partial p_2^{(m)}}{\partial x_1} - x_2\frac{\partial p_2^{(m)}}{\partial x_2}\right) \equiv 0.$$

Take

$$p_1^{(n)} = \sum_{j=0}^{n} a_j\, x_1^j\, x_2^{n-j}, \qquad p_2^{(m)} = \sum_{k=0}^{m} b_k\, x_1^k\, x_2^{m-k}.$$

Then it follows from (33) that

$$(34) \qquad \sum_{j=0}^{n}\sum_{k=0}^{m}[\,2\,(j-k)-(n-m)\,]\, a_j\, b_k\, x_1^{j+k}\, x^{(m+n)-(j+k)} \equiv 0.$$

Since not every $a_j = 0$ and not every $b_k = 0$ we can suppose that $a_{j_1} \neq 0$, $b_{k_1} \neq 0$. Hence we must have

$$2\,(j_1 - k_1) - (n - m) = 0,$$

and consequently $n - m$ is an even integer, say $m = n - 2r$. For any other value of j, say $j_s \neq j_1$ we have

$$2\,(j_s - k_1) - (n - m) \neq 0$$

and therefore $a_{j_s} = 0$. Likewise if $k_s \neq k_1$ we have also

$$2\,(j_1 - k_s) - (n - m) \neq 0$$

and therefore $b_{k_s} = 0$. Hence $p_1^{(n)}$ and $p_2^{(m)}$ have only one term each, and without loss of generality the coefficients of these terms can each be taken equal to unity. Furthermore we can suppose $m \neq n$. For if $m = n$ we could take the integral

$$H_1 = H - 1 = \frac{P_1}{P_2} - 1 = \frac{P_1 - P_2}{P_2},$$

for which $m \neq n$, and consequently $r \neq 0$.

We have then

$$p_1^{(n)} = x_1^{j_1}\, x_2^{n-j_1}, \qquad p_2^{(m)} = x^{j_1}\, x^{(n-j_1)}\,(x_1\, x_2)^{-r}.$$

Let us now make the transformation

$$x_1 = y_1 + f_1(y_1,\, y_2), \qquad x_2 = y_2 + f_2(y_1,\, y_2),$$

such that the transformed differential equations are (31). Then $P_1(x_1,\, x_2)$ becomes

$$S_1(y_1,\, y_2) = y_1^{j_1}\, y_2^{n-j_1} + s_1^{(n+1)} + s_1^{(n+2)} + \cdots,$$

and $P_2(x_1,\, x_2)$ becomes

$$S_2(y_1,\, y_2) = y_1^{j_1}\, y_2^{n-j_1}\,(y_1\, y_2)^{-r} + s_2^{(m+1)} + s_2^{(m+2)} + \cdots.$$

On differentiating

$$H = \frac{S_1(y_1 y_2)}{S_2(y_1 y_2)}$$

we have

$$S_2\left[\frac{\partial S_1}{\partial y_1}y_1' + \frac{\partial S_1}{\partial y_2}y_2'\right] - S_1\left[\frac{\partial S_2}{\partial y_1}y_1' + \frac{\partial S_2}{\partial y_2}y_2'\right] \equiv 0. \tag{35}$$

Consider the terms of degree $n + m + 1$ in the expression of (35),

$$s_2^{(m+1)}\left(y_1\frac{\partial s_1^{(n)}}{\partial y_1} - y_2\frac{\partial s_1^{(n)}}{\partial y_2}\right) - s_1^{(n+1)}\left(y_1\frac{\partial s_2^{(m)}}{\partial y_1} - y_2\frac{\partial s_2^{(m)}}{\partial y_2}\right) \equiv 0.$$

On taking

$$s_1^{(n+1)} = \sum_{k=0}^{n+1} c_k y_1^k y_2^{n+1-k}, \qquad s_2^{(m+1)} = \sum_{j=0}^{m+1} d_j y_1^j y_2^{m+1-j}$$

we find, after removing a common factor,

$$\sum_{k=0}^{n+1}[2(k-j_1)-1]c_k y_1^{k-r} y_2^{n+1-r-k} - \sum_{j=1}^{n+1-2r}[2(j-j_1)+2r-1]d_j y_1^j y^{n+1-j-2r}.$$

If r is positive the first $\sum$ contains more terms than the second, and if r is negative the second contains more than the first. Let us suppose $r > 0$ (the result is the same for $r < 0$). Then $c_k = 0$ for $k = 0, \cdots, r-1, n+2-r, \cdots, n+1$. If now we take $k - r = j$ for the remaining terms we get

$$\sum_{j=0}^{n+1-2r}[2(j-j_1+r)-1][c_j - d_j]y_1^j y_2^{n+1-2r-j} \equiv 0,$$

and therefore $c_j = d_j$, since $2(j - j_1 + r) - 1$ is an odd integer. Hence

$$s_1^{(n+1)} = (y_1 y_2)^r s_2^{(m+1)}. \tag{36}$$

From the terms of degree $m + n + 2$ we obtain

$$\begin{aligned} & s_2^{(m)}\left(y_1\frac{\partial s_1^{(n+2)}}{\partial y_1} - y_2\frac{\partial s_1^{(n+2)}}{\partial y_2}\right) - s_1^{(n)}\left(y_1\frac{\partial s_2^{(m+2)}}{\partial y_2} - y_2\frac{\partial s_2^{(m+2)}}{\partial y_2}\right) \\ & + s_2^{(m+1)}\left(y_1\frac{\partial s^{(n+1)}}{\partial y_1} - y_2\frac{\partial s^{(n+1)}}{\partial y_2}\right) - s_1^{(n+1)}\left(y_1\frac{\partial s_2^{(m+1)}}{\partial y_1} - y_2\frac{\partial s_2^{(m+1)}}{\partial y_2}\right) \\ & + s_2^{(m+2)}\left(y_1\frac{\partial s_1^{(n)}}{\partial y_1} - y_2\frac{\partial s_1^{(n)}}{\partial y_2}\right) - s_1^{(n+2)}\left(y_1\frac{\partial s_2^{(m)}}{\partial y_1} - y_2\frac{\partial s_2^{(m)}}{\partial y_2}\right) \\ & + r(Q_1^{(1)} + Q_2^{(1)})y_1^{2j_1-r+1}y_2^{2(n-j_1)-r+1} \equiv 0. \end{aligned} \tag{37}$$

The terms of the second line vanish by themselves. If we take

$$s_1^{(n+2)} = \sum_{k=0}^{n+2} e_k\, y_1^k\, y_2^{n+2-k}, \qquad s_2^{(m+2)} = \sum_{j=2}^{m+1} f_j\, y_1^j\, y_2^{m+2-j},$$

the identity (37) is easily reduced to

$$\sum_{j=0}^{n+2-2r} 2(j - j_1 + r - 1)(e_j - f_j)\, y_1^j\, y_2^{n+2-2r-j} \equiv - r\,(Q_1^{(1)} + Q_2^{(2)})\, y^{j_1-r+1}\, y_2^{n-j_1-r+1}.$$

For $j = j_1 - r + 1$ the coefficient of the left member vanishes, and the same must therefore be true of the right member. Therefore $Q_1^{(1)} + Q_2^{(2)} = 0$, since $r \neq 0$. For the other values of j we must have $e_j = f_j$. Hence

$$(38) \qquad s_1^{(n+2)} = (y_1 y_2)^r\, s_2^{(m+2)} + a y_1^{j_1+1}\, y_2^{n-j_1+1},$$

where a is some constant.

Proceeding in this manner it can be shown by induction that

$$(39) \quad Q_1^{(i)} + Q_2^{(i)} = 0 \quad i = 1, \cdots, \infty, \qquad \text{and} \qquad S_1 = (y_1 y_2)^r A\,(y_1 y_2)\, S_2,$$

where A is an ordinary power series in the product $(y_1 y_2)$, and this result is readily verified, for if we take

$$y_1' = y_1(+1 + Q^{(1)}\, y_1 y_2 + Q^{(2)}\, y_1^2 y_2^2 + \cdots),$$
$$y_2' = y_2(-1 - Q^{(1)}\, y_1 y_2 - Q^{(2)}\, y_1^2 y_2^2 - \cdots),$$
$$S_1 = AS_2,$$

where A is an arbitrary power series in $(y_1 y_2)$, and substitute in (35) it is found at once that this equation is satisfied.

It follows therefore if there exists an integral of the original differential equations of the form

$$H = \frac{P_1(x_1 x_2)}{P_2(x_1 x_2)},$$

where P_1 and P_2 are power series in x_1 and x_2, that a linear transcendental substitution which transforms the differential equations into the form (31) is such as to make $Q_1^{(i)} + Q_2^{(i)} = 0$, $i = 1, \cdots, \infty$, and furthermore transforms the quotient P_1/P_2 into a power series in the product $(y_1 y_2)$.

Returning now to equation (18) and putting

$$Q_1^{(1)} = - Q_2^{(1)} = Q, \qquad Q_1^{(2)} = - Q_2^{(2)} = 0,$$

it is clear that if $Q \neq 0$ the sum $(\alpha_1 + \beta_1)$ can be chosen so as to satisfy both equations. The difference $(\alpha_1 - \beta_1)$ remains arbitrary. From (18′) we have in general

$$Q_1^{(k+1)} = 0 = [(k-1)Q_1^{(1)} + kQ_2^{(1)}]\alpha_k \qquad Q_1^{(1)}\beta_k$$
$$+ \text{ known terms,}$$
$$Q_2^{(k+1)} = 0 = \qquad -Q_2^{(1)}\alpha_k + [k(Q_1^{(1)} + (k-1)Q_2^{(1)}]\beta_k$$
$$+ \text{ known terms.}$$

Since $Q_1^{(k+1)} + Q_2^{(k+1)} = 0$ the second equation is merely the negative of the first. Either of these equations gives

$$Q(\alpha_k + \beta_k) = \text{known terms.}$$

Thus $\alpha_k + \beta_k$ is determined by the choice $Q_1^{(k+1)} = Q_2^{(k+1)} = 0$. The difference $\alpha_k - \beta_k$ is undetermined. There are therefore infinitely many undetermined constants in the transformation, but whatever may be the value of these constants the reduced differential equations are

$$(40) \qquad \begin{aligned} y_1' &= +y_1(1 + Qy_1y_2), \\ y_2' &= -y_2(1 + Qy_1y_2). \end{aligned}$$

From these equations are derived the integral, $y_1 y_2 = c$, and therefore

$$(41) \qquad \begin{aligned} y_1 &= \sqrt{c}\, e^{(1+Qc)(t-t_0)}, \\ y_2 &= \sqrt{c}\, e^{-(1+Qc)(t-t_0)}. \end{aligned}$$

Proof of Convergence. In the substitution (2) we will denote the sum of all terms in x_1 which have the form $y_1(y_1y_2)^s$ by $y_1\phi_1$, and the sum of all terms in x_2 which have the form $y_2(y_1y_2)^s$ by $y_2\phi_2$. Then we can write

$$(42) \qquad \begin{aligned} x_1 &= y_1(1 + \phi_1) + f_1^*, \\ x_2 &= y_2(1 + \phi_2) + f_2^*; \end{aligned}$$

and from the process just described it is seen that the sum $\phi_1 + \phi_2$ is determined by the condition that the reduced equations shall have the form (40), while the difference $\phi_1 - \phi_2$ is undetermined.

The result of substituting (40) and (42) in the original equations (1) is

$$(43) \qquad \begin{aligned} y_1\frac{\partial f_1^*}{\partial y_1} - y_2\frac{\partial f_1^*}{\partial y_2} - f_1^* &= \frac{X_1 + Qy_1^2y_2(1+\phi_1) + Qy_1y_2f_1^*}{1 - Qy_1y_2} = R_1, \\ y_1\frac{\partial f_2^*}{\partial y_1} - y_2\frac{\partial f_2^*}{\partial y_2} - f_2^* &= -\frac{X_2 + Qy_1y_2^2(1+\phi_2) + Qy_1y_2f_2^*}{1 - Qy_1y_2} = R_2, \end{aligned}$$

since

$$y_1\frac{\partial(y_1\phi_1)}{\partial y_1} - y_2\frac{\partial(y_1\phi_1)}{\partial y_2} - (y_1\phi_1) \equiv 0,$$

$$y_1\frac{\partial(y_2\phi_2)}{\partial y_1} - y_2\frac{\partial(y_2\phi_2)}{\partial y_2} + (y_2\phi_2) \equiv 0.$$

Taking now

$$f_1{}^* = \sum\sum m_{ij}\, y_1^i\, y_2^j, \qquad i + j \geqq 2 \qquad \text{but} \qquad i \neq j + 1,$$
$$f_2{}^* = \sum\sum n_{ij}\, y_1^i\, y_2^j, \qquad i + j \geqq 2 \qquad \text{but} \qquad j \neq i + 1,$$

there results

$$(44) \qquad \begin{aligned} \sum\sum (i - j - 1)\, m_{ij}\, y_1^i\, y_2^j &= R_1, \\ \sum\sum (i - j + 1)\, n_{ij}\, y_1^i\, y_2^j &= R_2, \end{aligned}$$

from which the coefficients of the left members can be computed successively if Q is properly chosen.

If, however, we add to the first equation

$$\sum m_{i,\, i-1}\, y_1^i\, y_2^{i-1} = y_1(1 + \phi_1),$$

and subtract

$$\sum n_{i,\, i+1}\, y_1^i\, y_2^{i+1} = y_2(1 + \phi_2),$$

from the second equation we have

$$(45) \qquad \begin{aligned} \sum m_{i,\, i-1}\, y_1^i\, y_2^{i-1} + \sum{}' (i - j - 1)\, m_{ij}\, y_1^i\, y_2^j &= + y_1(1 + \phi_1) + R_1, \\ \sum n_{i,\, i+1}\, y_1^i\, y_2^{i+1} + \sum{}' (i - j + 1)\, n_{ij}\, y_1^i\, y_2^j &= - y_2(1 + \phi_2) - R_2. \end{aligned}$$

The coefficients of the left members of these equations (45) also can be determined so that the equations are satisfied identically whatever ϕ_1 and ϕ_2 in the right members may be, and the solutions thus obtained are convergent; for the solutions thus obtained are dominated by the solutions of the equations which are obtained by replacing each $(i - j + 1)$ in the left members of the first equation by $+1$ and each $(i - j + 1)$ of the second equation by -1 and taking all the coefficients in the expansions of R_1 and R_2 with the positive sign, and the solutions of the equations thus modified are known to be convergent.

The coefficients m_{ij} and n_{ij} thus obtained from (45) will be functions of ϕ_1 and ϕ_2. But we must have

$$(46) \qquad \begin{aligned} \sum m_{i,\, i-1}\, y_1^i\, y_2^{i-1} &= y_1(1 + \phi_1), \\ \sum n_{i,\, i+1}\, y_1^i\, y_2^{i+1} &= y_2(1 + \phi_2), \end{aligned}$$

the left members of which are convergent series. It will be observed that in (45) the factor $(1 + \phi_1)$ is always associated with y_1 except in the product $(y_1 y_2)$ and the factor $(1 + \phi_2)$ is associated in the same way with y_2. Hence the first equation of (46) contains $y_1(1 + \phi_1)$ as a factor and the second contains $y_2(1 + \phi_2)$ as a factor. Removing these factors and giving the m_{ij} and n_{ij} the values obtained from the solution of (45) we get

$$0 = [Q(1 + \phi_1)(1 + \phi_2) - Q]\, y_1 y_2 + [\,]\, y_1^2 y_2^2 + \cdots,$$
$$0 = [Q(1 + \phi_1)(1 + \phi_2) - Q]\, y_1 y_2 + [\,]\, y_1^2 y_2^2 + \cdots.$$

On removing the factor $y_1 y_2$ we have

$$
(47)\qquad \begin{aligned} 0 &= Q(\phi_1 + \phi_2) + [\,]\, y_1 y_2 + \cdots, \\ 0 &= Q(\phi_1 + \phi_2) + [\,]\, y_1 y_2 + \cdots. \end{aligned}
$$

These equations impose a condition only on $\phi_1 + \phi_2$ and they must therefore be identical. We have already seen that we can take $\phi_1 - \phi_2$ arbitrary. Hence we have

$$
(48)\qquad \begin{aligned} 0 &= Q(\phi_1 + \phi_2) + [\,]\, y_1 y_2 + \cdots, \\ 0 &= \phi_1 - \phi_2 + y_1 y_2 \psi(y_1 y_2) + \cdots. \end{aligned}
$$

If the arbitrary function ψ is given and is a convergent ordinary power series equations (48) admit a unique convergent solution for ϕ_1 and ϕ_2.

It might seem since m_{ij} and n_{ij} are functions of ϕ_1 and ϕ_2 that when the values thus derived for ϕ_1 and ϕ_2 are substituted in f_1^* and f_2^* they would then contain terms of the type $y_1^i y_2^{i-1}$, $y_1^i y_2^{i+1}$ respectively, but this is clearly not so, since the difference between the exponents of y_1 and y_2 in any term is not altered by multiplying the term by any power of $(y_1 y_2)$.

Example.—As an illustration of this method of solution let us consider the equation $d^2\phi/dt^2 + \sin\phi = 0$, whence $d^2\phi/dt^2 = -\phi + \frac{1}{6}\phi^3 - \frac{1}{120}\phi^5 + \cdots$. This equation will be reduced to the normal form if we take

$$
\phi = \tfrac{1}{2}(\xi_1 + i\xi_2), \qquad d\phi/dt = \tfrac{1}{2}(i\xi_1 + \xi_2),
$$

where $i = \sqrt{-1}$. The equations then are, if we take $it = \tau$,

$$
\xi_1' = \xi_1 - f, \qquad \xi_2' = -\xi_2 - if,
$$

where

$$
f = \tfrac{1}{48}(\xi_1 + i\xi_2)^3 - \tfrac{1}{3840}(\xi_1 + i\xi_2)^5 + \cdots.
$$

On making the linear-transcendental substitution

$$
\xi_1 = \eta_1 + \left[-\frac{1}{96}\eta_1^3 - \frac{3i}{128}\eta_1^2\eta_2 - \frac{1}{32}\eta_1\eta_2^2 - \frac{i}{192}\eta_2^3 \right] + \cdots,
$$

$$
\xi_2 = \eta_2 + \left[-\frac{i}{192}\eta_1^3 + \frac{1}{32}\eta_1^2\eta_2 - \frac{3i}{128}\eta_1\eta_2^2 + \frac{1}{96}\eta_2^3 \right] + \cdots,
$$

the reduced equations are found to be

$$
\eta_1' = \eta_1\left(1 - \frac{i}{16}\eta_1\eta_2\right), \qquad \eta_2' = -\eta_2\left(1 - \frac{i}{16}\eta_1\eta_2\right).
$$

The integral of these equations is $\eta_1\eta_2 = c$. It is convenient to take

$$
c = -16i\gamma.
$$

Then we find

$$\eta_1 = \sqrt{c}\, e^{(1-\gamma)\tau+\alpha}, \qquad \eta_2 = \sqrt{c}\, e^{-(1-\gamma)\tau-\alpha}.$$

If now we take $\alpha = -\pi i/4$ and replace τ by it we find

$$\phi = \tfrac{1}{2}(\xi_1 + i\xi_2) = [4 \sin (1-\gamma)t]\,\gamma^{1/2}$$
$$+ [\tfrac{1}{2} \sin (1-\gamma)t + \tfrac{1}{8} \sin 3(1-\gamma)t]\,\gamma^{3/2} + \cdots.$$

Since t does not enter explicitly in the original differential equation one can replace t by $(t - t_0)$ in the above solution in which $\phi = 0$ for $t = 0$.